Notes

Calendar

2025-10-01 – 2025-10-31						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

2025-11-01 – 2025-11-30						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

2025-12-01 – 2025-12-31						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

2026-01-01 – 2026-01-31						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

2026-02-01 – 2026-02-28						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	

2026-03-01 – 2026-03-31						
Mon	Tue	Wed	Thu	Fri	Sat	Sun
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

Days total:	168
Days past:	110
Days left:	58
Time left:	`duration(weeks: 8, days: 2)`
Percentage left:	0.34523809523809523

Abstract

read through Friedl Lück

- **Friedl Lück**
  -
- **Yi Liu:**
  1. Introduction
     - L^2-Alexander Torsion
       - $\tau^{(2)}$: Addmissible Triple $\to \frac{\mathbb{R}_{>0}\to\mathbb{R}_{\geq 0}}{\stackrel{\star}{=}}$

         $\tau^{(2)}(N,\varphi,\gamma)(t) = ???$
         - $f \stackrel{\star}{=} g \Leftrightarrow \exists r \in \mathbb{R} \forall t \in \mathbb{R}_{>0} : f(t) = t^r g(t)$
         - "getwistete" variante der full L^2-Alexander Torsion
       - N : 3-Mannifold
         - needed: (empty-/incompressible toral-) boundary
         - in [1]: prime
       - Addmissible Triple := (

         $N,$

         $\varphi : H^1(N,\partial N;\mathbb{R}) \stackrel{[1]}{=} \pi_1(N) \xrightarrow[\mathrm{Grp}]{} \mathbb{R},$

         $\gamma : \pi_1(N) \xrightarrow[\mathrm{Grp}]{} G,$

         h_1 : Prop $= (\exists\tilde{\varphi} : \pi_1(N) \xrightarrow[\mathrm{Grp}]{\gamma} G \xrightarrow[\mathrm{Grp}]{\tilde{\varphi}} \mathbb{R} = \pi_1(N) \xrightarrow[\mathrm{Grp}]{\varphi} \mathbb{R})$
     - full L^2-Alexander Torsion with respect to φ
       - $\tau^{(2)}(N,\varphi)(t)$

- if $\gamma = \mathrm{id}_{\pi_1(N)}$
- Zu zeigen:
  - $\tau^{(2)}(N, \varphi)(t)$
    - continuous
    - positive
    - asymptotically monomial in both ends
      - $f : \mathbb{R}_{>0} \to [0, +\infty)$

$$\exists C_{+\infty}, d_{+\infty} \in \mathbb{N} : \quad \lim_{t\to+\infty} \frac{f(t)}{t^{d_{+\infty}}} = C_{+\infty} \text{ monomial in the limit } t \to +\infty$$

$$\exists C_{0+}, d_{0+} \in \mathbb{N} : \quad \lim_{t\to0+} \frac{f(t)}{t^{d_{0+}}} = C_{0+} \text{ monomial in the limit } t \to 0+$$ [1]

  - $\deg \tau^{(2)}(N, \varphi)(t)$ is Thurston norm of φ Definition of deg?
    - $\deg^{\mathrm{a}}(f) = d_{+\infty} - d_{0+}$
      - examples:
        - $\frac{t^2}{t-1} \overset{\deg^{\mathrm{a}}}{\mapsto} 2 - 1 = 0$
      - $\frac{e^{-t}}{t^p} \overset{\deg^{\mathrm{a}}}{\mapsto} 0 - \underbrace{(-p)}_{\lim_{t\to0+} \frac{\frac{\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}}{t^p}}{t^{d_{0+}}}} = p$
- $\tau^{(2)}(N, \varphi, \gamma : \pi_1(N) \to G)(t)$
  - continuous
  - positive (if G *residually finite* & (N, γ) *weakly acyclic*)
  - $\underbrace{\widetilde{\deg}}_{\text{generalized degree}} \tau^{(2)}(N, \varphi, \gamma : \pi_1(N) \to G)(t)$ bounded by *Thurston norm* of φ
    - $\deg^{\mathrm{b}}_{+\infty}(f) = \inf\left\{D_{+\infty} \in \mathbb{R} : \lim_{t\to+\infty} \frac{f(t)}{t^{D_{+\infty}}} = 0\right\}$
    - $\deg^{\mathrm{b}}_{0+}(f) = \sup\left\{D_{0+} \in \mathbb{R} : \lim_{t\to0+} \frac{f(t)}{t^{D_{0+}}} = 0\right\}$
    - $\deg^{\mathrm{b}}(f) = \deg^{\mathrm{b}}_{+\infty}(f) - \deg^{\mathrm{b}}_{0+}(f)$ *growth bound degree*[1]

Chapters (Liu)

1. Introduction
2. Preliminaries
3. Regular Fugledge-Kadison determinant
4. Multiplicatively convex functions
5. Multiplicative convexity and exponent bound
6. Continuity of degree
7. asymptotics for integral matrices
8. L^2-Alexander torsion of 3-manifolds
9. examples

Chapters (Masters thesis)

1. torsion invariants
   1. Alexander polynomial (Knots)
   2. Reidemeister torsion $\rho(X)$ / Alexander torsion $\tau(K)$

Unterschied?

   3. L^2-Torsion invariants
      1. twisting (see: *full* $\rightsquigarrow$ L^2-Alexander Torsion)
2. thurston norm
3. connection
   1. asymptotic degree

[1]Definition 1.3 aus Yi Liu

Reidemeister torsion

Definition the **determinant** is defined as::

- A a free R-module of rank n
- $T : A \underset{R\text{-mod}}{\to} A$
  - $\rightsquigarrow$

$$\bigwedge^n T : \bigwedge^n A \underset{R\text{-mod}}{\to} \bigwedge^n A$$

$\det : \{n : \mathbb{N}\} \to \{A : R\text{-mod}\} \to \{\operatorname{rank} A = n\} \to$

$$\left(A \underset{R\text{-mod}}{\to} A\right) \underset{R\text{-mod}}{\to} R$$

$\det(T) = r$

$$\text{where } \forall v_1, ..., v_n \in A : \left(\bigwedge^n T\right)(v_1 \wedge ... \wedge v_n) = r \cdot (v_1 \wedge ... \wedge v_n)$$

Definition Chain complexes are defined as::

$(C_*, \partial_*) : \mathrm{Chain}_{R\text{-mod}}$

$$\cdots \xleftarrow{\partial_{-2}} C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

$$\forall i \in \mathbb{Z} : \partial_{i+1} \circ \partial_i = 0$$

The requirement $\partial_{i+1} \circ \partial_i = 0$ makes writing the chain complex in the following way "obvious":

$$\begin{tikzcd}
\cdots & C_{-2} \arrow[l, dotted, "0"'] \arrow[dl, "\partial_{-2}"'] & C_0 \arrow[l, dotted, "0"'] \arrow[dl, "\partial_0"'] & C_2 \arrow[l, dotted, "0"'] \arrow[dl, "\partial_2"'] & \cdots \arrow[l, dotted, "0"'] \arrow[dl, "\partial_4"'] \\
\cdots & C_{-1} \arrow[l, dotted, "0"'] \arrow[u, "\partial_{-1}"'] & C_1 \arrow[l, dotted, "0"'] \arrow[u, "\partial_1"'] & C_3 \arrow[l, dotted, "0"'] \arrow[u, "\partial_3"'] & \cdots \arrow[l, dotted, "0"']
\end{tikzcd}$$

Definition We define the **Reidemeister-Torsion** as::

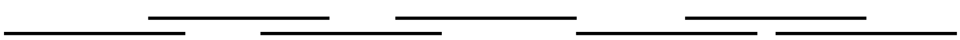
- $C_* : \mathrm{Chain}_{R\text{-mod}}$
  - $\forall i : C_i$ is a free R-module of finite rank n_i
- $C_i = 0$ for almost all $i \in \mathbb{Z}$
- C_* *contractible*

$\rho : \{\forall i : \operatorname{rank} C_i < \infty\} \to (C_* : \mathrm{Chain}_{R\text{-mod}}) \to R$

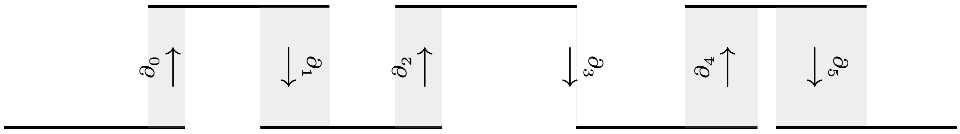
ρ

Beer coaster trick

For intuition we will think of each chain module as a beer coaster. We will lay them next to each other as in the picture below:



The even ones are on the bottom, and the odd ones on the top. How much the coasters overlap corresponds to how much the differentials transport from one module onto the next.



Defintion

ok to assume start at 0?

- $G \in \mathrm{Grp}$
- finite $\mathbb{Z}G$ chain complex $C_0(\tilde{K};V) \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_m} C_m(\tilde{K};V)$
  - $C_*(\tilde{K};V) = V \otimes_{\mathbb{Z}G} C_*(\tilde{K})$
    - $V \cong \mathbb{R}^n$ with representation $\rho : \mathbb{Z}G \curvearrowright V$

please provide some examples

- $$\begin{tikzcd} C_{2\bullet}(\tilde{K};V) := & C_0(\tilde{K};V) & \oplus & C_2(\tilde{K};V) \arrow[dll, "\partial_2"'] & \oplus & C_4(\tilde{K};V) \arrow[dll, dotted, "\partial_4"'] & \cdots \\ C_{2\bullet+1}(\tilde{K};V) := & C_1(\tilde{K};V) \arrow[u, "\partial_1"'] & \oplus & C_3(\tilde{K};V) \arrow[u, "\partial_3"'] & \oplus & \cdots \end{tikzcd}$$

there is an isormorphism between $C_{2\bullet+1}$ and $C_{2\bullet}$, if the chain is acyclic, meaning each homology group is 0, or equivalently the coasters overlap completely

make this align, each line must be of varying sizes

beliebig?

Definition[2]

- K compact CW-complex
  - $\chi(K) = 0$
  - $n \in \mathbb{N}$
  - $\rho : \pi_1 K \underset{\mathrm{Grp}}{\rightarrow} \mathrm{SL}_n(\mathbb{C})$ representation
- $\tilde{K}$ universal cover
  - $\pi_1 K \curvearrowright \tilde{K}$ deck transform
  - $C_*(\tilde{K};\mathbb{Z})$ singular chain complex

twisted chain complex:

$$C_*(\tilde{K};\rho) := C_*(\tilde{K};\mathbb{Z}) \otimes_\rho \mathbb{C}^n$$

$$= \frac{C_*(\tilde{K};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^n}{\cancel{\underbrace{v}_{C_*(\tilde{K};\mathbb{Z})} \otimes \underbrace{\lambda^{-1}}_{\pi_1 K} \underbrace{w}_{\mathbb{C}^n} = \rho(\lambda) v \otimes w}}$$

ergibt doch keinen Sinn, dass $\underbrace{\lambda^{-1}}_{\pi_1 K} \underbrace{w}_{\mathbb{C}^n}$, weil das ja nicht in $\mathbb{C}^n$ ist?

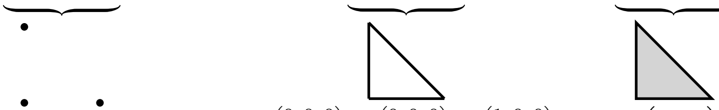
$$= \frac{C_*(\tilde{K};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^n}{\cancel{\lambda^{-1} v \otimes w = v \otimes \rho(\lambda) w}}$$

ich vermute, dass es so sein sollte?

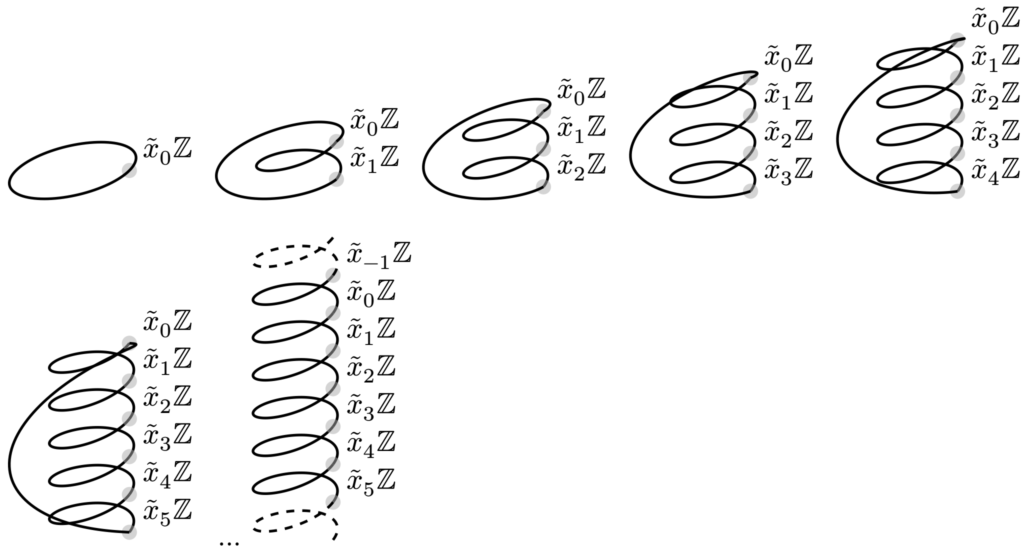
But where does λ^{-1} live in? Or: what is $C_*(\tilde{K};\mathbb{Z})$? Apparently $\lambda^{-1} \in \pi_1(K)$, but why?

[2] https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion

$$C_*(K;\mathbb{Z}) = \underbrace{C_0(K;\mathbb{Z})}_{\mathbb{Z}(0\ 0\ 0)\oplus\mathbb{Z}(1\ 0\ 0)\oplus\mathbb{Z}(0\ 1\ 0)} \xleftarrow{\partial_1} \underbrace{C_1(K;\mathbb{Z})}_{\mathbb{Z}\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}\oplus\mathbb{Z}\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix}\oplus\mathbb{Z}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}} \xleftarrow{\partial_2} \underbrace{C_2(K;\mathbb{Z})}_{\mathbb{Z}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}} \xleftarrow{\partial_3} \cdots$$



For $\tilde{K}$, we would have $\#C_i(\tilde{K};\mathbb{Z}) = \infty$ for most i.



This makes[3] $\underbrace{C_0(\tilde{K};\mathbb{Z})}_{\cong\mathbb{Z}^{\pi_1 K}} \xleftarrow{\partial_1} \underbrace{C_1(\tilde{K};\mathbb{Z})}_{\cong\mathbb{Z}^{\pi_1 K}} \xleftarrow{\partial_2} 0$

[3]https://en.wikipedia.org/wiki/Exponential_object

Twisted tensor products

We have:

wie besser aufschreiben?

$$C_*^{(2)}\left(\tilde{X};\kappa\right) : \mathrm{Chain}\ \left(\text{Hilbert-Mod}_{\mathrm{FinGen}}\right)$$

$\otimes_{\kappa(\varphi,t)}$?

$$C_*^{(2)}\left(\tilde{X};\kappa\right) := \ell^2\pi \otimes_{\kappa(\varphi,t)} C_*\left(\tilde{X}\right)$$

We need

$$\begin{aligned}\otimes &: \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \left(\left\{\pi \underset{\mathrm{Grp}}{\to} \mathbb{R}\right\} \to \mathbb{R}_{>0} \to \left\{\mathbb{Z}\pi \underset{\mathrm{Ring}}{\to} \mathbb{R}\pi\right\}\right) \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}}\end{aligned}$$

for $\kappa(\varphi,\gamma,t)(g)$ or

$$\begin{aligned}\otimes &: \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \left(\left\{\pi \underset{\mathrm{Grp}}{\to} \mathbb{R}\right\} \to \left\{\pi \underset{\mathrm{Grp}}{\to} G\right\} \to \mathbb{R}_{>0} \to \left\{\mathbb{Z}\pi \underset{\mathrm{Ring}}{\to} \mathbb{R}\pi\right\}\right) \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}}\end{aligned}$$

for $\kappa(\varphi,t)(g)$. Because from context we are provided with φ, γ and t we need

$$\begin{aligned}\otimes &: \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \left\{\mathbb{Z}\pi \underset{\mathrm{Ring}}{\to} \mathbb{R}\pi\right\} \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}} \\ &\to \text{Hilbert-Mod}_{\mathrm{FinGen}}\end{aligned}$$

and from this, we have[4]:

$$\otimes_{(\cdot)} : (\text{mod-}R) \to \left(L \underset{\mathrm{Ring}}{\to} R\right) \to (L\text{-mod}) \underset{\mathrm{Ab}}{\to} \mathrm{Ab}$$

$$A \otimes_\kappa B = \frac{A \otimes B}{v \otimes \lambda^{-1} w = \kappa(\lambda) v \otimes w}$$

the "twisted tensor" product.

actually $\rho : \pi_1 K \to \mathrm{SL}_{n(\mathbb{C})}$ in the article[5]

fix this in my definition!

[4]https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion
[5]https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion

Mainly used in *Reidemeister-Torsion* for chain-complexes:

$$C_*(N, \kappa) := C_*\left(\tilde{N}, \mathbb{Z}\right) \otimes_\kappa \mathbb{C}^n$$

with

$$\begin{aligned} \otimes : \left(C_*\left(\tilde{N}, \mathbb{Z}\right) : \text{mod-}\mathbb{Z}\right) \\ \to \left(\mathbb{C} \underset{\mathbb{Z}\text{-mod}}{\to} C_*\left(\tilde{N}, \mathbb{Z}\right)\right) \\ \to (\mathbb{C} : \mathbb{Z}\text{-mod}) \\ \underset{\mathbb{Z}\text{-mod}}{\to} \mathbb{Z}\text{-mod} \end{aligned}$$

$$v \otimes_\kappa \lambda^{-1} w = \kappa(\lambda) v \otimes w$$

Example:

$$\mathbb{R}^3 \otimes_\rho \left(\mathbb{R}^*\right)^3$$

$$\rho((a, b, c)) = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes_\rho (a, b, c) =$$

ℓ^2**-alexander torsion**

Definition

why?

- X connected, finite CW-complex
  - connected: because otherwise $\pi_1(N)$ is not well-defined (only for $(X, \cdot)$)
  - finite: finitely many equivariant cells
    - equivariant G-cells: $G/H \times D^n$ is glued into the skeleton $(H \lhd G)$
- $\pi = \pi_1(X)$
- $\varphi \in H^1(X; \mathbb{R}) \cong \left\{\pi \underset{\text{Grp}}{\to} \mathbb{R}\right\}$
- $\kappa : \left\{\pi \underset{\text{Grp}}{\to} \mathbb{R}\right\} \to \mathbb{R}_{>0} \to \left\{\mathbb{Z}\pi \underset{\text{Ring}}{\to} \mathbb{R}\pi\right\}$

  $\kappa(\varphi, t)(g) = t^{\varphi(g)} g$

$\boldsymbol{\kappa(\varphi, t)}$**-twisted** $\boldsymbol{\ell^2}$**-chain complex:**

wie besser aufschreiben?

$$C_*^{(2)}(\tilde{X}; \kappa) : \text{Chain}\ (\text{Hilbert-Mod}_{\text{FinGen}})$$

$\otimes_{\kappa(\varphi,t)}$?

$$C_*^{(2)}(\tilde{X}; \kappa) := \ell^2\pi \otimes_{\kappa(\varphi,t)} C_*(\tilde{X})$$

Definition

- only if $C_*^{(2)}(\tilde{X}; \kappa)$ is of *determinant class*
- pick *cellular basis* of $\tilde{X}$

cellular basis?

$\boldsymbol{\ell^2}$**-torsion**:

$$\rho^{(2)}\left(C_*^{(2)}\right) := \sum_{n \in \mathbb{Z}} (-1)^{n+1} \log \det_{\mathcal{R}(G)} d_n^{(2)}$$

Definition

- only if $C_*^{(2)}(\tilde{X}; \kappa)$ is ℓ^2-acylcic (no reduced homology)

was bedeutet das?

full $\boldsymbol{\ell^2}$**-Alexander torsion**

$$\begin{aligned}
\tau^{(2)}(X, \varphi)(t) &:= \exp\left(-\rho^{(2)}\left(C_*^{(2)}(\tilde{X}; \kappa)\right)\right) \\
&= \exp\left(-\sum_{n \in \mathbb{Z}} (-1)^{n+1}\right) \log \det_{\mathcal{R}(G)} d_n^{(2)}) \\
&= \exp\left(\sum_{n \in \mathbb{Z}} (-1)^n \log \det_{\mathcal{R}(G)} d_n^{(2)}\right) \\
&= \prod_{n \in \mathbb{Z}} \exp\left((-1)^n \log \det_{\mathcal{R}(G)} d_n^{(2)}\right) \\
&= \prod_{n \in \mathbb{Z}} \left(\det_{\mathcal{R}(G)} d_n^{(2)}\right)^{(-1)^n} \\
&= \dots \det_{\mathcal{R}(G)} d_{-4}^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_{-3}^{(2)}} \det_{\mathcal{R}(G)} d_{-2}^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_{-1}^{(2)}} \det_{\mathcal{R}(G)} d_0^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_1^{(2)}} \det_{\mathcal{R}(G)} d_2^{(2)} \dots
\end{aligned}$$

base change matrix?

Proposition

- picking another *cellular basis*

generalized permutaion matriy?

$\Rightarrow$ new *base change matrix* is *generalized permutaion matriy* P with entries $\pm t^{\varphi(g_i)} g_i$

Proposition Fuglede-Kadison determinant of P (for $t \neq 1$) is

stimmt das?

$$t^{\varphi(g_1)+\ldots+\varphi(g_{k_n})}$$

woher kommt hier k_n?

Proposition

consists = $\bigcup \mathbb{T}_i$

- N connected, compact irreducible 3-manifold
- $\#\pi_1(N) = \infty$
- ∂N empty or consists of incompressible tori

incompressible tori?

Definition

- "twist"
- go from $\tilde{N}$ to ? using $\gamma : \pi \underset{\mathrm{Grp}}{\to} G$ with

are we working with different coverings now?

is $G \lhd \pi$? currently only working with universal cover $\tilde{N}$, so maybe work with N_G?

$$\begin{tikzcd}
\pi \arrow[r, "\gamma"] \arrow[dr, "\varphi"] & G \arrow[d, dashed] \\
& \mathbb{R}
\end{tikzcd}$$

ℓ^2-Alexander torsion

$$\tau^{(2)}(N, \gamma, \varphi) = \exp\left(-\rho^{(2)}\left(C_*^{(2)}\left(\tilde{X}; \kappa\right)\right)\right)$$

$$\text{where } \kappa(\varphi, \gamma, t)(g) := t^{\varphi(g)}\gamma(g)$$

Bibliography

[1] J. Dubois, S. Friedl, and W. Lück, “The L^2-Alexander Torsion of 3-Manifolds.” Accessed: Jan. 14, 2026. [Online]. Available: http://arxiv.org/abs/1410.6918